Fourier Representation of Signals and LTI Systems.

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3.0 Fourier Representation of Signals and LTI Systems.

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3.1 Introduction.

✓ Signals are represented as superposition's of complex sinusoids which leads to a useful expression for the system output and provide a characterization of signals and systems.

✓ Example in music, the orchestra is a superposition of sounds generated by different equipment having different frequency range such as string, base, violin and et c. The same example applied the choir team.

✓ Study of signals and systems using sinusoidal representation is termed as Fourier Analysis introduced by Joseph Fourier (1768-1830).

✓ There are four distinct Fourier representations, each applicable to different class of signals.
There are four distinct Fourier representations, class of signals:

(i) Periodic Signals.
(ii) Nonperiodic Signals.
(iii) Discrete-Time Periodic Signals.
(iv) Continuous-Time Periodic Signals.

Fourier Series (FS) applies to continuous-time periodic signals.
Discrete Time Fourier Series (DTFS) applies to discrete-time periodic signals.
Nonperiodic signals have Fourier Transform (FT) representation.
Discrete Time Fourier Transform (DTFT) applies to a signal that is discrete in time and non-periodic.
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(1) Discrete Time Periodic Signal

(2) Continuous Time Periodic Signal

(3) Continuous Time Non Periodic Signal

(4) Discrete Time Non Periodic Signal

Fourier Representation
3.2.1 Periodic Signals: Fourier Series Representation

- If \( x[n] \) is a **discrete-time signal** with fundamental period \( N \) then \( x[n] \) of DTFS is represents as,

\[
\hat{x}[n] = \sum_{k} A[k] e^{j k \Omega_o n}
\]

- where \( \Omega_o = \frac{2\pi}{N} \) is the fundamental frequency of \( x[n] \).
- The frequency of the \( k \)th sinusoid in the superposition is \( k \Omega_o \).
If \( x(t) \) is a \textbf{continuous-time signal} with fundamental period \( T \), \( x(t) \) of FS is represents as

\[
\hat{x}(t) = \sum_{k} A[k] e^{jk\omega_0 t}
\]

where \( \omega_0 = 2\pi/T \) is the fundamental frequency of \( x(t) \). The frequency of the \( k \)th sinusoid is \( k\omega_0 \) and each sinusoid has a common period \( T \).

A sinusoid whose frequency is an integer multiple of a fundamental frequency is said to be a \textbf{harmonic} of a sinusoid at the fundamental frequency. For example \( e^{jk\omega_0 t} \) is the \( k \)th harmonic of \( e^{j\omega_0 t} \).

The variable \( k \) indexes the frequency of the sinusoids, \( A[k] \) is the function of frequency.
3.2.2 Nonperiodic Signals: Fourier-Transform Representation.

☑ The Fourier Transform representations employ complex sinusoids having a continuum of frequencies.

☑ The signal is represented as a weighted integral of complex sinusoids where the variable of integration is the sinusoid’s frequency.

☑ Continuous-time sinusoids are used to represent continuous signal in

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \]

where \( X(j\omega)/(2\pi) \) is the “weight” or coefficient applied to a sinusoid of frequency \( \omega \) in the FT representation.
Discrete-time sinusoids are used to represent discrete time signal in DTFT.

It is unique only a $2\pi$ interval of frequency. The sinusoidal frequencies are within $2\pi$ interval.

\[
\hat{x}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n} d\omega
\]
3.3 Discrete Time Periodic Signals: Discrete Time Fourier Series.

- The Discrete Time Fourier Series representation:

\[
x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_o n}
\]

- where \( x[n] \) is a periodic signal with period \( N \) and fundamental frequency \( \Omega_o = 2\pi/N \). \( X[k] \) is the DTFS coefficient of signal \( x[n] \).

- The relationship of the above equation,

\[
X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\Omega_o n}
\]
Given \( N \) value of \( x[n] \) we can find \( X[k] \).

Given \( N \) value of \( X[k] \) we can find \( x[n] \) vise-versa.

The \( X[k] \) is the **frequency-domain representation** of \( x[n] \).

DTFS is the only Fourier representation that can be numerically evaluated using computer, e.g. used in numerical signal analysis.
Example 3.1: Determining DTFS Coefficients.

Find the frequency-domain representation of the signal in Figure 3.2 below.

Solution:

Step 1: Determine $N$ and $\Omega_0$.

The signal has period $N=5$, so $\Omega_0=2\pi/5$.

Also the signal has odd symmetry, so we sum over $n = -2$ to $n = 2$ from equation...
Step 2: Solve for the frequency-domain, $X[k]$.

From step 1, we found the fundamental frequency, $N = 5$, and we sum over $n = -2$ to $n = 2$.

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\Omega_0 n}$$

$$X[k] = \frac{1}{5} \sum_{n=-2}^{2} x[n] e^{-jk2\pi n/5}$$

$$= \frac{1}{5} \left\{ x[-2] e^{jk4\pi/5} + x[-1] e^{jk2\pi/5} + x[0] e^{j0} + x[1] e^{-jk2\pi/5} + x[2] e^{-jk4\pi/5} \right\}$$
Cont’d…

From the value of $x\{n\}$ we get,

$$X[k] = \frac{1}{5} \left\{ 1 + \frac{1}{2} e^{j k 2\pi / 5} - \frac{1}{2} e^{-j k 2\pi / 5} \right\}$$

$$= \frac{1}{5} \left\{ 1 + j \sin \left( k \frac{2\pi}{5} \right) \right\}$$

**Step 3: Plot the magnitude and phase of DTFS.**

From the equation, one period of the DTFS coefficient $X[k]$, $k=-2$ to $k=2$, in the rectangular and polar coordinate as

$$X[-2] = \frac{1}{5} - j \frac{\sin \left( \frac{4\pi}{5} \right)}{5} = 0.232 e^{-j 0.531}$$

$$X[-1] = \frac{1}{5} - j \frac{\sin \left( \frac{2\pi}{5} \right)}{5} = 0.276 e^{-j 0.760}$$
The above figure shows the magnitude and phase of $X[k]$ as a function of frequency index $k$.

**Figure 3.3: Magnitude and phase of the DTFS coefficients for the signal in Fig. 3.2.**
Cont’d…

Just to Compare, for different range of $n$.
Calculate $X[k]$ using $n=0$ to $n=4$ for the limit of the sum.

$$X[k] = \frac{1}{5} \left\{ x[0]e^{j0} + [1]e^{-j2\pi/5} + [2]e^{-jk4\pi/5} + [3]e^{-jk6\pi/5} + [4]e^{-jk8\pi/5} \right\}$$

$$X[k] = \frac{1}{5} \left\{ 1 - \frac{1}{2} e^{-jk2\pi/5} + \frac{1}{2} e^{-jk8\pi/5} \right\}$$

This expression is different from which we obtain from using $n=-2$ to $n=2$. Note that,

$$e^{-jk8\pi/5} = e^{-jk2\pi} e^{jk2\pi/5} = e^{jk2\pi/5}$$

$n=-2$ to $n=2$ and $n=0$ to $n=4$, yield equivalent expression for the DTFS coefficient.
3.4 Continuous-Time Periodic Signals: Fourier Series.

✓ Continues time periodic signals are represented by Fourier series (FS). The fundamental period is \( T \) and fundamental frequency \( \omega_0 = 2\pi/T \).

\[
x(t) = \sum_{k=-\infty}^{\infty} X(k) e^{jk\omega_0 t}
\]

✓ \( X[k] \) is the FS coefficient of signal \( x(t) \).

\[
X(k) = \frac{1}{T} \int_{0}^{T} x(t) e^{-jk\omega_0 t} dt
\]

✓ Below is the relationship of the above equation,

\[
x(t) \xleftarrow{\text{FS; } \omega_0} X(k)
\]
The FS coefficient $X(k)$ is known as frequency-domain representation of $x(t)$ because each FS coefficient is associated with a complex sinusoid of a different frequency.

$k$ determines the frequency of complex sinusoid associated with $X(k)$. 

Cont’d…
Example 3.2: Direct Calculation of FS Coefficients.

Determine the FS coefficients for the signal $x(t)$ depicted in Figure 3.4.

Solution:

Step 1: Determine $T$ and $\omega_0$.

The period of $x(t)$ is $T=2$, so $\omega_0=2\pi/2 = \pi$. On the interval $0 \leq t \leq 2$, one period of $x(t)$ is expressed as $x(t)=e^{-2t}$, so it yields

$$X(k) = \frac{1}{2} \int_{0}^{2} e^{-(2+j\pi \omega_0)t} dt$$

Step 2: Solve for $X[k]$.

$$X(k) = \frac{1}{T} \int_{0}^{T} x(t)e^{-j\omega_0 t} dt$$
Cont’d…

**Step 3: Plot the magnitude and phase spectrum.**

\[
X(k) = \frac{-1}{2(2 + jk\pi)} e^{-(2 + jk\pi)t} \bigg|_0^2 \\
X(k) = \frac{-1}{4 + 2jk\pi} \left(e^{-4} e^{-jk2\pi} - 1\right) \\
X(k) = \frac{1 - e^{-4}}{4 + jk2\pi}
\]

- From the above equation for example \( k=0 \),
  \[ X(k) = (1 - e^{-4})/4 = 0.245. \]

- Since \( e^{-jk2\pi}=1 \) Figure 3.5 shows the magnitude spectrum \(|X(k)|\) and the phase spectrum \( \arg\{X(k)\} \).
Figure 3.5: Magnitude and Phase Spectra.
3.5 Complex Sinusoidal and Frequency Response of LTI System.

✓ The response of an LTI system to a sinusoidal input that lead to the characterization of the system behavior that is called frequency response of the system.

✓ An impulse response $h[n]$ and the complex sinusoidal input $x[n] = e^{j\Omega n}$. 

Cont’d…

**Discrete-Time (DT)**

Derivation:

\[
y[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]
\]

\[
y[n] = \sum_{k=-\infty}^{\infty} h[k] e^{j\Omega(n-k)}
\]

factor \(e^{j\Omega n}\) out

\[
y[n] = e^{j\Omega n} \sum_{k=-\infty}^{\infty} h[k] e^{-j\Omega k}
\]

\[
= H(e^{j\Omega}) e^{j\Omega n}
\]

- Frequency Response:

\[
H(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\Omega k}
\]
The output of a complex sinusoidal input to an LTI system is a complex sinusoid of the same frequency as the input, multiplied by the frequency response of the system.
Example 3.3: Frequency Response.
The impulse response of a system is given as
\[ h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t) \]
Find the expression for the frequency response, and plot the magnitude and phase response.

**Solution:**

**Step 1: Find frequency response, \( H(j\omega) \).**

Substitute \( h(t) \) into equation below,

\[
H(j\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau
\]

\[
H(j\omega) = \frac{1}{RC} \int_{-\infty}^{\infty} \frac{\tau}{RC} e^{\frac{-\tau}{RC}} u(\tau) e^{-j\omega\tau} d\tau
\]

\[
= \frac{1}{RC} \int_{0}^{\infty} e^{-\left(j\omega + \frac{1}{RC}\right)\tau} d\tau
\]

\[
= \frac{1}{RC} \left[ -\frac{1}{j\omega + \frac{1}{RC}} \right] e^{-\left(j\omega + \frac{1}{RC}\right)\tau} \bigg|_{0}^{\infty}
\]

\[
= \frac{1}{RC} \left( -\frac{1}{j\omega + \frac{1}{RC}} \right) (0 - 1)
\]

\[
= \frac{1}{RC} \left( j\omega + \frac{1}{RC} \right)
\]
Cont’d…

**Step 2:** From frequency response, get the magnitude & phase response.

*The magnitude response* is,

\[ |H(j\omega)| = \frac{1}{RC} \sqrt{\omega^2 + \left(\frac{1}{RC}\right)^2} \]
Figure 3.1: Frequency response of the \( RC \) circuit (a) Magnitude response. (b) Phase response.

*The phase response* is \( \arg\{H(j\omega)\} = -\arctan(\omega RC) \)
3.6 Discrete-Time Non Periodic Signals: Discrete-Time Fourier Transform

The DTFT representation of time domain signal,

\[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega})e^{jn\Omega} \, d\Omega \]

\( X[k] \) is the DTFT of the signal \( x[n] \).

Below is the relationship of the above equation,

\[ X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\Omega} \]

\( x[n] \xrightarrow{DTFT} X[e^{j\Omega}] \)
The FT, $X[j\omega]$, is known as the **frequency-domain representation** of the signal $x(t)$.

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}) e^{j\Omega n} d\Omega$$

The above equation is the **inverse FT**, where it maps the frequency domain representation $X[j\omega]$ back to the time domain.
3.7 Continuous-Time Nonperiodic Signals: Fourier Transform.

The Fourier transform (FT) is used to represent a continuous time nonperiodic signal as a superposition of complex sinusoids.

FT representation,

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega \]

where,

\[ X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} \]

Below is the relationship of the above equation,
Example 3.4: FT of a Real Decaying Exponential.
Find the Fourier Transform (FT) of $x(t) = e^{-at} u(t)$.

Solution:
The FT does not converge for $a \leq 0$, since $x(t)$ is not absolutely integrable, that is

$$\int_{0}^{\infty} e^{-at} dt = \infty, \quad a \leq 0$$

for $a > 0$, we have

$$X(j\omega) = \int_{0}^{\infty} e^{-at} u(t) e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t} dt$$

$$= -\frac{1}{a+j\omega} e^{-(a+j\omega)t} \bigg|_{0}^{\infty}$$

$$= -\frac{1}{a+j\omega}$$

$$= \frac{1}{a+j\omega}$$
Cont’d…

Converting to polar form, we find that the magnitude and phase of $X(j\omega)$ are respectively given by

$$|X(j\omega)| = \frac{1}{\left(a^2 + \omega^2\right)^{1/2}}$$

and

$$\arg\{X(j\omega)\} = -\arctan\left(\frac{\omega}{a}\right)$$

This is shown in Figure 3.6 (b) and (c).
Figure 3.6: (a) Real time-domain exponential signal. (b) Magnitude spectrum. (c) Phase spectrum.
3.8 Properties of Fourier Representations.

- The **four Fourier representations** discussed in this chapter are summarized in Table 3.2.
- Attached Appendix C is the comprehensive table of all properties.
## Table 3.2: Fourier Representation.

<table>
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<th>Time Domain</th>
<th>Periodic (t,n)</th>
<th>Non periodic (t,n)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Continuous (t)</strong></td>
<td><strong>Fourier Series</strong></td>
<td><strong>Fourier Transform</strong></td>
</tr>
<tr>
<td>[ x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t} ]</td>
<td>[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} , dw ]</td>
<td></td>
</tr>
<tr>
<td>[ X[k] = \frac{1}{T} \int_{0}^{T} x(t) e^{-jk\omega_0 t} , dt ]</td>
<td>[ X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} , dt ]</td>
<td></td>
</tr>
<tr>
<td>[ x(t) \text{ has period} \ \omega_0 = \frac{2\pi}{T} ]</td>
<td>[ ]</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Discrete (t)</strong></th>
<th><strong>Discrete Time Fourier Series</strong></th>
<th><strong>Non periodic (k,w)</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>[ x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_0 n} ]</td>
<td>[ ]</td>
<td>[ ]</td>
</tr>
<tr>
<td>[ X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\Omega_0 n} ]</td>
<td>[ ]</td>
<td>[ ]</td>
</tr>
<tr>
<td>[ x[n] \text{ and } X[k] \text{ have period} \ N ]</td>
<td>[ ]</td>
<td>[ ]</td>
</tr>
<tr>
<td>[ \Omega_0 = \frac{2\pi}{n} ]</td>
<td>[ ]</td>
<td>[ ]</td>
</tr>
</tbody>
</table>
3.9 Linearity and Symmetry Properties.

- All four Fourier representations involve linear operations.

- The linearity property is used to find Fourier representations of signals that are constructed as sums of signals whose representation are already known.
Example 3.5: Linearity in the Fourier Series.

Given \( z(t) \) which is the periodic signal. Use the linearity property to determine the FS coefficients \( Z[k] \).

**Solution:**

From Example 3.31 (Text) we have

\[
x(t) \xleftarrow{FS; 2\pi} X[k] = \frac{1}{k\pi} \sin \left( \frac{k\pi}{4} \right)
\]

\[
y(t) \xleftarrow{FS; 2\pi} Y[k] = \frac{1}{k\pi} \sin \left( \frac{k\pi}{2} \right)
\]

The linearity property implies that

\[
z(t) \xleftarrow{FS; 2\pi} Z[k] = \frac{3}{2k\pi} \sin \left( \frac{k\pi}{4} \right) + \frac{1}{2k\pi} \sin \left( \frac{k\pi}{2} \right)
\]
### 3.9.1 Symmetry Properties: Real and Imaginary Signal.

Table 3.3: Symmetry Properties for Fourier Representation of Real-and Imaginary–Valued Time Signals.

<table>
<thead>
<tr>
<th>Representation</th>
<th>Real-Valued Time Signal</th>
<th>Imaginary-Valued Time Signal</th>
</tr>
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<tbody>
<tr>
<td>FT</td>
<td>$X^*(j\omega) = X(-j\omega)$</td>
<td>$X^*(j\omega) = -X(-j\omega)$</td>
</tr>
<tr>
<td>FS</td>
<td>$X^*[k] = X[-k]$</td>
<td>$X^*[k] = -X[-k]$</td>
</tr>
<tr>
<td>DTFT</td>
<td>$X^*(e^{j\omega}) = X(e^{-j\omega})$</td>
<td>$X^*(e^{j\omega}) = -X(e^{-j\omega})$</td>
</tr>
<tr>
<td>DTFS</td>
<td>$X^*[k] = X[-k]$</td>
<td>$X^*[k] = -X[-k]$</td>
</tr>
</tbody>
</table>
3.9.2 Symmetry Properties: Even and Odd Signal.

\[ X^*(j\omega) = \int_{-\infty}^{\infty} x(-t)e^{-j\omega t} \, dt \]

- Change of variable \( \tau = -t \).

\[ X^*(j\omega) = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega \tau} \, d\tau \]

- The only way that the condition \( X^*(j\omega) = X(j\omega) \) holds is for the imaginary part of \( X(j\omega) \) to be zero.
Cont’d…

☑ If time signal is **real and even**, then the frequency-domain representation of it also **real**.

☑ If time signal is **real and odd**, then the frequency-domain representation is **imaginary**.
The convolution of the signals in the time domain transforms to multiplication of their respective Fourier representations in the frequency domain.

The convolution property is a consequence of complex sinusoids being eigen functions of LTI system.

3.10.1 Convolution of Non periodic Signals.
3.10.2 Filtering.
3.10.3 Convolution of Periodic Signals.
3.10.1 Convolution of Non Periodic Signals.

Convolution of two non periodic continuous-time signals $x(t)$ and $h(t)$ is defined as

$$ y(t) = h(t) * x(t) $$

$$ = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau. $$

express $x(t-\tau)$ in term of FT:
Cont’d…

Substitute into convolution integral yields,

\[ y(t) = \int_{-\infty}^{\infty} h(\tau) \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} e^{-j\omega \tau} d\omega \right] d\tau \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} h(\tau)e^{-j\omega \tau} d\tau \right] X(j\omega)e^{j\omega t} d\omega \]
✓ Convolution of $h(t)$ and $x(t)$ in the time domain corresponds to **multiplication** of Fourier transforms, $H(j\omega)$ and $X(j\omega)$ in the **frequency domain** that is:

$$y(t) = h(t) * x(t) \xrightarrow{FT} Y(j\omega) = X(j\omega)H(j\omega).$$

✓ **Discrete-time** nonperiodic signals, if

$$x[n] \xrightarrow{DTFT} X(e^{j\Omega}) \quad \text{and} \quad h[n] \xrightarrow{DTFT} H(e^{j\Omega}) \quad \text{then} \quad y[n] = h[n] * x[n] \xrightarrow{DTFT} Y(e^{j\Omega}) = X(e^{j\Omega})H(e^{j\Omega}).$$
Example 3.6: Convolution in Frequency Domain.

Let \( x(t) = (1/(\pi t)) \sin(\pi t) \) be the input of the system with impulse response \( h(t) = (1/(\pi t)) \sin(2\pi t) \). Find the output \( y(t) = x(t) * h(t) \).

**Solution:**

From Example 3.26 (text, Simon) we have,

\[
\begin{align*}
\mathcal{F}\{x(t)\} & = X(j\omega) = \begin{cases} 1, & |\omega| < \pi \\ 0, & |\omega| > \pi \end{cases} \\
\mathcal{F}\{h(t)\} & = H(j\omega) = \begin{cases} 1, & |\omega| < 2\pi \\ 0, & |\omega| > 2\pi \end{cases}
\end{align*}
\]

Since

\[
y(t) = x(t) * h(t) \iff \mathcal{F}\{y(t)\} = Y(j\omega) = X(j\omega)H(j\omega)
\]

it follows that

\[
Y(j\omega) = \begin{cases} 1, & |\omega| < \pi \\ 0, & |\omega| > \pi \end{cases}
\]

We conclude that \( y(t) = \left(\frac{1}{\pi t}\right) \sin(\pi t) \).